Perturbative and Non-Perturbative Aspects of $\mathcal{N} = 8$ Supergravity

Sergio Ferrara[1,2] and Alessio Marrani[1]

1 *Physics Department,Theory Unit, CERN,*
CH 1211, Geneva 23, Switzerland
sergio.ferrara@cern.ch
Alessio.Marrani@cern.ch

2 *INFN - Laboratori Nazionali di Frascati,*
Via Enrico Fermi 40,00044 Frascati, Italy

Abstract

Some aspects of quantum properties of $\mathcal{N} = 8$ supergravity in four dimensions are discussed for non-practitioners.

At perturbative level, they include the Weyl trace anomaly as well as composite duality anomalies, the latter being relevant for perturbative finiteness. At non-perturbative level, we briefly review some facts about extremal black holes, their Bekenstein-Hawking entropy and attractor flows for single- and two- centered solutions.

1 Lecture I
On "Quantum" $\mathcal{N} = 8$, $d = 4$ Supergravity

$\mathcal{N} = 8$, $d = 4$ "quantum" supergravity may be defined by starting with the Einstein-Hilbert action, and setting "perturbative" Feynman rules as a *bona fide* gauge theory of gauge particles of spin 2, the gravitons. In supersymmetric gravity theories with $\mathcal{N}$-extended supersymmetry in $d = 4$ space-time dimensions, the massless particle content is given by

$$\binom{\mathcal{N}}{k} \equiv \frac{\mathcal{N}!}{k!\,(\mathcal{N}-k)!} \text{ particles of } \textit{helicity } \lambda = 2 - \frac{k}{2}, \tag{1.1}$$

where $k_{\max} = \mathcal{N}$, and $\mathcal{N} \leqslant 8$ if $|\lambda| \leqslant 2$ is requested (namely, no higher spin fields in the massless spectrum).

One possible approach to "quantum" supergravity is to consider it as it comes from M-theory restricted to the massless sector. The problem is that this theory, even if preserving maximal $\mathcal{N} = 8$ supersymmetry in $d = 4$ space time dimensions (corresponding to $32 = 8 \times 4$ supersymmetries), is *not* uniquely defined, because of the multiple choice of internal compactification manifolds and corresponding duality relations:

$$\textbf{I}. \; M_{11} \longrightarrow M_4 \times T_7 \qquad (GL^+(7, \mathbb{R}) \text{ and } SO\,(7) \text{ manifest});$$

$$\textbf{II}. \; M_{11} \longrightarrow AdS_4 \times S^7 \qquad (SO\,(8) \text{ manifest, } \textit{gauged}); \tag{1.2}$$

$$\textbf{III}. \; M_{11} \longrightarrow M_4 \times T_{7,\mathcal{R}} \qquad (SL(8, \mathbb{R}) \text{ and } SO\,(8) \text{ manifest}),$$

where T_7 is the 7-torus and S^7 is the 7-sphere. $T_{7,\mathcal{R}}$ denotes the case in which, according to Cremmer and Julia [1], the dualization of 21 vectors and 7 two-forms makes $SL(8, \mathbb{R})$ (in which $GL^+(7, \mathbb{R})$ is maximally embedded) manifest as maximal non-compact symmetry of the Lagrangian. Note that in case **III** one can further make $E_{7(7)}$ (and its maximal compact subgroup $SU\,(8)$) manifest *on-shell*, by exploiting a Cayley transformation supplemented by a rotation through $SO\,(8)$ gamma matrices on the vector 2-form field strengths [1, 2]. As we discuss further below, $E_{7(7)}$ can be promoted to a Lagrangian symmetry if one gives up manifest diffeomorphism invariance, as given by treatment in [3], then used in the anomaly study of [4].

It is worth remarking that $\mathcal{N} = 8$, $d = 4$ *gauged* supergravity with gauge group $SO(8)$ cannot be used for electroweak and strong interactions model building, because

$$SO\,(8) \not\supseteq SU\,(3) \times SU\,(2) \times U\,(1). \tag{1.3}$$

Furthermore, also the cosmological term problem arises out: the vacuum energy in anti De Sitter space AdS_4 is much higher than the vacuum energy in Standard Model of non-gravitational interactions (see *e.g.* the discussion in [5]). However, by exploiting the AdS_4/CFT_3 correspondence, theory **II** of (1.2) recently found application in $d = 3$ condensed matter physics (see *e.g.* [6] for a review and list of Refs.). Furthermore, the recently established fluid-gravity correspondence was object of many studies (see *e.g.* [7] for recent reviews and lists of Refs.).

The fundamental massless fields (and the related number $\sharp$ of degrees of freedom) of M-theory in $d = 11$ flat space-time dimensions are

$$g_{\mu\nu} \; (\textit{graviton}) : \qquad \sharp = \frac{(d-1)(d-2)}{2} - 1, \qquad \text{in } d = 11 : \sharp = 44;$$

$$\Psi_{\mu\alpha} \; (\textit{gravitino}) : \qquad \sharp = (d-3)2^{(d-3)/2}, \qquad \text{in } d = 11 : \sharp = 128; \tag{1.4}$$

$$A_{\mu\nu\rho} \; (\textit{three-form}) : \quad \sharp = \frac{(d-2)(d-3)(d-4)}{3!}, \qquad \text{in } d = 11 : \sharp = 84.$$

Because a $(p+1)$-form ("Maxwell-like" gauge field) A_{p+1} couples to p-dimensional extended objects, and its "magnetic" dual B_{d-p-3} couples to $(d-p-4)$-dimensional extended objects, it follows that the fundamental (massive) objects acting as sources of the theory are $M2$- and $M5$-branes.

In general, a compactification on an n-torus T_n has maximal manifest non-compact symmetry $GL^+(n,\mathbb{R}) \sim \mathbb{R}^+ \times SL(n,\mathbb{R})$. The metric g_{IJ} of T_n parametrizes the $n(n+1)/2$-dimensional coset $\mathbb{R}^+ \times \frac{SL(n,\mathbb{R})}{SO(n)}$, whereas the Kaluza-Klein vectors $g_{\widehat{\mu}}^I$ are in the $\mathbf{n}'$ irrep. of $GL^+(n,\mathbb{R})$ itself. By reducing M-theory on T_7 a $d=4$ theory with maximal ($\mathcal{N}=8$) local supersymmetry arises. By splitting the $d=11$ space-time index $\mu = 0,1,...,10$ as $\mu = (\widehat{\mu}, I)$, where $\widehat{\mu} = 0,1,...3$ is the $d=4$ space-time index, and $I = 1,...,7$ is the internal manifold index, the bosonic degrees of freedom of M-theory split as follows (below (1.6), for simplicity's sake we will then refrain from hatting the $d=4$ curved indices):

$$
g_{\mu\nu} \longrightarrow \begin{cases} g_{\widehat{\mu}\widehat{\nu}} \ (d=4 \ graviton), & \mathbf{1}+\mathbf{1}; \\ \\ g_{\widehat{\mu}}^I \ (vectors), & \mathbf{7}'; \\ \\ g_{IJ} \ (scalars), & \mathbf{28}; \end{cases} \tag{1.5}
$$

$$
A_{\mu\nu\rho} \longrightarrow \begin{cases} A_{\widehat{\mu}\widehat{\nu}\widehat{\rho}} \ (d=4 \ domain \ wall), & \sharp = 0; \\ \\ A_{\widehat{\mu}\widehat{\nu}I} \ (antisymmetric \ tensors : strings), & \mathbf{7}; \\ \\ A_{\widehat{\mu}IJ} \ (vectors), & \mathbf{21}; \\ \\ A_{IJK} \ (scalars), & \mathbf{35}, \end{cases} \tag{1.6}
$$

where the indicated irreps. pertain to the maximal manifest non-compact symmetry $GL^+(7,\mathbb{R})$, whose maximal compact subgroup is $SO(7)$. The 28 scalars g_{IJ} (metric of T_7) parametrize the coset $\mathbb{R}^+ \times \frac{SL(7,\mathbb{R})}{SO(7)}$.

By switching to formulation **III** of (1.2) [1], the 7 antisymmetric rank-2 tensors $A_{\widehat{\mu}\widehat{\nu}I}$ (sitting in the $\mathbf{7}$ of $GL^+(7,\mathbb{R})$ can be dualized to scalars ϕ^I (in the $\mathbf{7}'$ of $GL^+(7,\mathbb{R})$), and therefore one obtains $35 + 28 + 7 = 70$ scalar fields. It is worth remarking that in Cremmer and Julia's [1] theory the gravitinos ψ_I and the gauginos χ_{IJK} respectively have the following group theoretical assignment[1] (I in $\mathbf{8}$ of $SU(8)$):

$$
\text{theory } \mathbf{III} \text{ [1]} : \begin{cases} \psi_I : \ \underset{\mathbf{8}}{SO(7)} \subset \underset{\mathbf{8}_s}{SO(8)} \subset \underset{\mathbf{8}}{SU(8)}; \\ \\ \chi_{IJK} : \ \underset{\mathbf{8}+\mathbf{48}}{SO(7)} \subset \underset{\mathbf{56}_s}{SO(8)} \subset \underset{\mathbf{56}}{SU(8)}. \end{cases} \tag{1.7}
$$

Thus, in this theory the 70 scalars arrange as

$$
\text{theory } \mathbf{III} \text{ [1]} : s = 0 \ dofs : \underset{(\sharp=70)}{SO(7)} \underset{\mathbf{1}+\mathbf{7}+\mathbf{21}+\mathbf{35}}{} \subset \underset{\mathbf{35}_v+\mathbf{35}_c}{SO(8)} \subset \underset{\mathbf{70}}{SU(8)}, \tag{1.8}
$$

where $\mathbf{70}$ is the rank-4 completely antisymmetric irrep. of $SU(8)$, the maximal compact subgroup of the U-duality group $E_{7(7)}$ (also called $\mathcal{R}$-symmetry).

---

[1]As evident from (1.7), we use a different convention with respect to [5] (see *e.g.* Table 36 therein). Indeed, we denote as $\mathbf{8}_v$ of $SO(8)$ the irrep. which decomposes into $\mathbf{7}+\mathbf{1}$ of $SO(7)$, whereas the two spinorial irreps. $\mathbf{8}_s$ and $\mathbf{8}_c$ both decompose into $\mathbf{8}$ of $SO(7)$. The same change of notation holds for $\mathbf{35}$ and $\mathbf{56}$ irreps..

On the other hand, also the vector fields $A_{\widehat{\mu}IJ}$ (sitting in the $\mathbf{21}$ of $GL^+(7,\mathbb{R})$) can be dualized to $A_{\widehat{\mu}}^{IJ}$ (sitting in the $\mathbf{21'}$ of $GL^+(7,\mathbb{R})$). Together with $g_{\widehat{\mu}}^I$, the "electric" and "magnetic" vector degrees of freedom can thus be arranged as follows:

$$
\begin{array}{c}
s = 1 \ dofs : \\
{\scriptstyle(\sharp=56)}
\end{array}
\left\{
\begin{array}{l}
GL^+(7,\mathbb{R}) \subset SL(8,\mathbb{R}) \subset E_{7(7)}; \\
{\scriptstyle \mathbf{7'}+\mathbf{21'}+\mathbf{7}+\mathbf{21} \quad \mathbf{28'}+\mathbf{28} \quad \mathbf{56}} \\[1em]
SO(7) \ \subset SO(8) \subset SU(8). \\
{\scriptstyle \mathbf{7}+\mathbf{21}+\mathbf{7}+\mathbf{21} \quad \mathbf{28}+\mathbf{28} \quad \mathbf{28}+\overline{\mathbf{28}}}
\end{array}
\right.
\tag{1.9}
$$

The counting of degrees of freedom is completely different in the *gauged* maximal supergravity theory **II** of (1.2), based on the $AdS_4 \times S^7$ solution of $d = 11$, $\mathcal{N} = 1$ M-theory field equations; in this framework, rather than using torus indices as in theories **I** and **III** of (1.2), Killing vector/spinor techniques are used (for a discussion, see *e.g.* [8], and the lectures [9], and Refs. therein). However, the 70 scalars still decompose[2] as $\mathbf{35}_v + \mathbf{35}_c$ of $SO(8)$ but, with respect to the chain of branchings (1.8), they lack of any $SO(7)$ interpretation. It is worth recalling that a formulation of this theory directly in $d = 4$ yields to the de Wit and Nicolai's $\mathcal{N} = 8$, $d = 4$ *gauged* supergravity [10].

Since the 70 scalar fields fit into an unique irrep. of $SU(8)$, it follows that they parameterize a non-compact coset manifold $\frac{G}{SU(8)}$. Indeed, the $SU(8)$ under which both the scalar fields and the fermion fields transform is the *"local"* $SU(8)$, namely the stabilizer of the scalar manifold. On the other hand, the $SU(8)$ appearing in the second line of (1.9), *i.e.* the one under which the vector 2-form self-dual/anti-self-dual field strengths transform, is the *"global"* $SU(8)$ ($\mathcal{R}$-symmetry group). Roughly speaking, the physically relevant group $SU(8)$ is the diagonal one in the product $SU_{\mathrm{local}}(8) \times SU_{\mathrm{global}}(8)$ (see also discussion below).

Remarkably, there exists an *unique* simple, non-compact Lie group with real dimension $70 + 63 = 133$ and which embeds $SU(8)$ as its maximal compact subgroup: this is the real, non-compact split form $E_{7(7)}$ of the exceptional Lie group E_7, thus giving rise to the symmetric, rank-7 coset space

$$
\frac{E_{7(7)}}{SU(8)/\mathbb{Z}_2},
\tag{1.10}
$$

which is the scalar manifold of $\mathcal{N} = 8$, $d = 4$ supergravity ($\mathbb{Z}_2$ is the kernel of the $SU(8)$-representations of even rank; in general, spinors transform according to the double cover of the stabilizer of the scalar manifold; see *e.g.* [11, 12]).

$E_{7(7)}$ acts as electric-magnetic duality symmetry group [13], and its maximal compact subgroup $SU(8)$ has a chiral action on fermionic as well as on (the vector part of the) bosonic fields. While the chiral action of $SU(8)$ on fermions directly follows from the chirality (complex nature) of the relevant irreps. of $SU(8)$ (as given by Eq. (1.7)), the chiral action on vectors is a crucial consequence of the electric-magnetic duality in $d = 4$ space-time dimensions. Indeed, this latter allows for "self-dual/anti-self-dual" complex combinations of the field strengths, which can then fit into complex irreps. of the stabilizer H of the coset scalar manifold G/H itself. For the case of maximal $\mathcal{N} = 8$ supergravity, the relevant chiral complex irrep. of $H = SU(8)$ is the rank-2 antisymmetric $\mathbf{28}$, as given by Eq. (1.9).

Note that if one restricts to the $SL(8,\mathbb{R})$-covariant sector, the chirality of the action of electric-magnetic duality is spoiled, because the maximal compact subgroup of $SL(8,\mathbb{R})$, namely $SO(8)$, has not chiral irreps.

---

[2]There are three distinct 35-dimensional $SO(8)$ irreps., usually denoted as $\mathbf{35}_v$, $\mathbf{35}_s$ and $\mathbf{35}_c$, obeying the relations:

$$
(ab) \leftrightarrow [ABCD]_+ \, ; \quad [abcd]_+ \leftrightarrow [ABCD]_- \, ; \quad [abcd]_- \leftrightarrow (AB) \, ,
$$

where $a, b = 1, ..., 8$ are in $\mathbf{8}_v$, $A, B, C, D = 1, ..., 8$ are in $\mathbf{8}_s$ (or in $\mathbf{8}_c$), and "$\pm$" denotes self-dual/anti-self-dual irreps.. For a discussion, see *e.g.* [1] and [9].

Composite (sigma model G/H) anomalies can arise in theories in which G has a maximal compact subgroup with a *chiral* action on bosons and/or fermions (see *e.g.* [14, 15, 4]). Surprising cancellations among the various contributions to the composite anomaly can occur as well. An example is provided by $\mathcal{N} = 8$, $d = 4$ supergravity itself, in which standard anomaly formulæ yield the remarkable result [15, 4]

$$3Tr_{\mathbf{8}}X^3 - 2Tr_{\mathbf{28}}X^3 + Tr_{\mathbf{56}}X^3 = (3 - 8 + 5)\,Tr_{\mathbf{8}}X^3 = 0, \tag{1.11}$$

where X is any generator of the Lie algebra $\mathfrak{su}(8)$ of the *rigid* (*i.e. global*) $SU(8)$ group ($\mathcal{R}$-symmetry). In light of the previous considerations, the first and third contributions to (1.11) are due to fermions: the 8 gravitinos ψ_A and the 56 spin-$\frac{1}{2}$ fermions χ_{ABC}, respectively, whereas the second contribution is due to the 28 chiral vectors. Note that, for the very same reason, the *local* $SU(8)$ (stabilizer of the non linear sigma-model of scalar fields), under which only fermions do transform[3], would be *anomalous* [14]. In an analogous way, in [15] it was discovered that $\mathcal{N} = 6$ and $\mathcal{N} = 5$ "pure" supergravities are *composite anomaly-free*, whereas $\mathcal{N} \leqslant 4$ theories are not.

A crucial equivalence holds at the homotopical level:

$$E_{7(7)} \cong (SU(8)/\mathbb{Z}_2) \times \mathbb{R}^{70}, \tag{1.12}$$

implying that the two group manifolds have the same De Rham cohomology. This is a key result, recently used in [4] to show that the aforementioned absence of $SU(8)$ current anomalies yield to the absence of anomalies for the non-linearly realized $E_{7(7)}$ symmetry, thus implying that the $E_{7(7)}$ continuous symmetry of classical $\mathcal{N} = 8$, $d = 4$ supergravity is preserved at all orders in perturbation theory (see *e.g.* [16, 17, 18, 19, 20, 21, 22, 23]). This implies the perturbative finiteness of supergravity at least up to seven loops; Bern, Dixon *et al.* explicitly checked the finiteness up to four loops included [16] (computations at five loops, which might be conclusive, are currently in progress; for a recent review, see *e.g.* [24]).

In order to achieve the aforementioned result on the anomalies of $E_{7(7)}$, in [4] the manifestly $E_{7(7)}$-covariant Lagrangian formulation of $\mathcal{N} = 8$, $d = 4$ supergravity [3] was exploited, by using the ADM decomposition of the $d = 4$ metric, namely[4]:

$$g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}\left(dx^i + N^i dt\right)\left(dx^j + N^j dt\right), \tag{1.13}$$

with lapse N and shift N^i (h_{ij} is the metric on the spatial slice). Within this approach [25, 3], the diffeomorphism symmetry is not realized in the standard way on the vector fields: the 28 vector fields A_μ^Λ of the original formulation [1, 10] are replaced by 56 vector fields $A_i^{\mathbb{B}}$ ($\mathbb{B} = 1, ..., 56$) with only spatial components, which recover the number of physical degrees of freedom by switching to an Hamiltonian formulation. Besides the 56×56 symplectic metric Ω:

$$\Omega^T = -\Omega,\ \Omega^2 = -\mathbb{I}, \tag{1.14}$$

a crucial quantity is the scalar field-dependent 56×56 symmetric matrix $\mathcal{M}$ (see Eq. (2.20) below), which is symplectic (see *e.g.* [26]):

$$\mathcal{M}\Omega\mathcal{M} = \Omega, \tag{1.15}$$

---

[3]Also scalar fields transform under *local* $SU(8)$, but they do not contribute to the composite anomaly, because they sit in the *self-real* (and thus *non-chiral*) rank-4 antisymmetric irrep. **70** of $SU(8)$.

[4]We use units in which the Newton gravitational constant G and the speed of light in vacuum c are all put equal to 1.

and negative definite due to the positivity of the vector kinetic terms (see also discussion below). $\mathcal{M}$ allows for the introduction of a symplectic, scalar field-dependent complex structure:

$$\mathcal{J} \equiv \mathcal{M}\Omega \Rightarrow \mathcal{J}^2 = \mathcal{M}\Omega\mathcal{M}\Omega = -\mathbb{I}. \qquad (1.16)$$

Thus, the equations of motion of the 56 vector fields $A_i^{\mathbb{B}}$ can be expressed as a twisted self-duality condition[5] [1] for their super-covariant fields strengths $\hat{F}_{\mu\nu}^{\mathbb{A}}$, namely (see [3, 4] for further elucidation)

$$\hat{F}_{\mu\nu}^{\mathbb{A}} = -\frac{1}{2\sqrt{|g|}}\epsilon_{\mu\nu}{}^{\rho\sigma}\mathcal{J}_{\mathbb{B}}^{\mathbb{A}}\hat{F}_{\mu\nu}^{\mathbb{B}}. \qquad (1.17)$$

Although the time components $A_0^{\mathbb{B}}$ do not enter the Lagrangian, they appear when solving the equations of motion for the spatial components $A_i^{\mathbb{B}}$, and diffeomorphism covariance is recovered on the solutions of the equations of motion [3, 4].

From power counting arguments in quantum gravity, an n-loop counterterm contains $2n+2$ derivatives, arranged such that it does not vanish *on-shell*. In $\mathcal{N}=8$ supergravity the first (non-BPS) full superspace integral which is $E_{7(7)}$-invariant is the *super-Vielbein* superdeterminant, which may contain as last component a term $\sim \partial^8 R^4$ (see *e.g.* [28], and also [29]), then possibly contributing to a divergence in the four-graviton amplitude. However, in [22] R. Kallosh argued that, by exploiting the light-cone formulation, candidate counterterms can be written in chiral, but *not* in real, light-cone superspace. This would then imply the ultraviolet finiteness of $\mathcal{N}=8$, $d=4$ supergravity, *if* supersymmetry and $E_{7(7)}$ symmetry are non-anomalous. Recently, in [30] the latter symmetry was advocated by the same author to imply ultraviolet finiteness of the theory to all orders in perturbation theory.

A puzzling aspect of these arguments is that string theory certainly violates continuous $E_{7(7)}$ symmetry at the perturbative level, as it can be easily realized by considering the dilaton dependence of loop amplitudes (see *e.g.* [23]). However, this is not the case for $\mathcal{N}=8$ supergravity. From this perspective, two (perturbatively finite) theories of quantum gravity would exist, with 32 local supersymmetries; expectedly, they would differ at least in their non-perturbative sectors, probed *e.g.* by black hole solutions. String theorists [31, 32, 33] claim that $\mathcal{N}=8$, $d=4$ supergravity theory is probably not consistent at the non-perturbative level. From a purely $d=4$ point of view, their arguments could be overcome by excluding from the spectrum, as suggested in [18], black hole states which turn out to be singular or ill defined if interpreted as purely four-dimensional gravitational objects. Inclusion of such singular states (such as $\frac{1}{4}$-BPS and $\frac{1}{2}$-BPS black holes) would then open up extra dimensions, with the meaning that a non-perturbative completion of $\mathcal{N}=8$ supergravity would lead to string theory [31]. Extremal black holes with a consistent $d=4$ interpretation may be defined as having a Bertotti-Robinson [34] $AdS_2 \times S^2$ near-horizon geometry, with a non-vanishing area of the event horizon. In $\mathcal{N}=8$ supergravity, these black holes are[6] $\frac{1}{8}$-BPS or non-BPS (for a recent review and a list of Refs., see *e.g.* [35]). The existence of such states would in any case break the $E_{7(7)}(\mathbb{R})$ continuous symmetry, because of Dirac-Schwinger-Zwanziger dyonic charge quantization conditions. The breaking of $E_{7(7)}(\mathbb{R})$ into an arithmetic subgroup $E_{7(7)}(\mathbb{Z})$ would then manifest only in exponentially suppressed contributions to perturbative amplitudes (see *e.g.* the discussion in [4], and Refs. therein), in a similar way to instanton effects in non-Abelian gauge theories.

The composite anomaly concerns the gauge-scalar sector of the supergravity theories. Another anomaly, originated in the gravitational part of the action is the so-called *gravitational anomaly*, which only counts the basic degrees of freedom associated to the field content of the

---

[5]For interesting recent developments on twisted self-duality, see [27].
[6]We also remark that these are the only black holes for which the *Freudenthal duality* [36, 37] is well defined.

theory itself [38, 39] (see also [40] for a review):

$$g_{\mu\nu} \langle T^{\mu\nu} \rangle_{1-loop} = \frac{\mathcal{A}}{32\pi^2} \int d^4x \sqrt{|g|} \left(R^2_{\mu\nu\lambda\rho} - 4R^2_{\mu\nu} + R^2 \right), \qquad (1.18)$$

where $\langle T^{\mu\nu} \rangle_{1-loop}$ is the 1-loop *vev* of the gravitational stress-energy tensor. In general, this trace anomaly is a total derivative and therefore it can be non-vanishing only on topologically non-trivial $d = 4$ backgrounds. Furthermore, as found long time ago by Faddeev and Popov [41], $(p + 1)$-form gauge fields have a complicated quantization procedure, due to the presence of ghosts; thus, their contribution to the parameter $\mathcal{A}$ appearing in the formula (1.18) vary greatly depending on the field under consideration. This is because at the quantum level different field representations are generally inequivalent [38]. Consequently, one may expect that different formulations of $\mathcal{N} = 8$, $d = 4$ supergravity (1.2), give rise to different gravitational anomalies. This is actually what happens:

- in the formulation **III** of (1.2) [1], with maximal manifest compact symmetry $SO\,(8)$, the antisymmetric tensors $A_{\mu\nu I}$ are dualized to scalars, and $\mathcal{A} \neq 0$.

- in the formulation **I** of (1.2) [1], with maximal manifest compact symmetry $SO\,(7)$, obtained by compactifying $d = 11$ M-theory on T_7, the antisymmetric tensors $A_{\mu\nu I}$ are not dualized, and, as found some time ago in [38], the gravitational anomaly vanishes: $\mathcal{A} = 0$. Recently, a wide class of models has been shown to have $\mathcal{A} = 0$, by exploiting *generalized mirror symmetry* for seven-manifolds [42].

- in the formulation **II** of (1.2) [10, 39] (see also [8, 9] and the discussion above), with maximal manifest compact *gauged* symmetry $SO\,(8)$, the gravitational anomaly is the sum of two contributions: one given by (1.18), and another one related to the non-vanishing cosmological constant Λ, given by

$$\frac{\mathcal{B}}{12\pi^2} \int d^4x \sqrt{|g|} \Lambda^2, \qquad (1.19)$$

where $\mathcal{B}$, through the relation $\Lambda \sim -e^2$ [39], vanishes whenever the charge e normalization beta function[7] [43]

$$\beta_e\,(s) = \frac{\hbar}{96\pi^2} e^3 C_s \left(1 - 12s^2 \right) (-1)^{2s} \qquad (1.20)$$

vanishes, namely in $\mathcal{N} > 4$ supergravities (compare *e.g.* Table II of [39] with Table 1 of [43]). The contribution to the coefficients $\mathcal{A}$ and $\mathcal{B}$ of (1.18) and (1.19) depends on the spin s of the *massless* particle, but also, as mentioned above, on the its field representation ([39]; see also *e.g.* Table 1 of [42]):

$s :$	$0\ (\phi)$	$0\ (A_{\mu\nu\rho})$	$\frac{1}{2}$	$1\ (A_\mu)$	$1\ (A_{\mu\nu})$	$\frac{3}{2}$	2
$360\mathcal{A} :$	4	-720	7	-52	364	-233	848
$60\mathcal{B} :$	-1	0	-3	-12	0	137	$-522.$

$$\qquad (1.21)$$

---

[7]C_s is the appropriate (positive) *quadratic invariant* for the gauge group representation in which the particle of spin s sits (see *e.g.* Table 1 of [43], and Refs. therein).

2 Lecture II
(Multi-Center) Black Holes and Attractors

If $E_{7(7)}$ is a continuous non-anomalous symmetry of $\mathcal{N} = 8$ supergravity, then it is likely that non-perturbative effects are exponentially suppressed in perturbative amplitudes.

Black holes (BHs) are examples of non-perturbative states which, in presence of Dirac-Schwinger-Zwanziger dyonic charge quantization, would break $E_{7(7)}(\mathbb{R})$ to a suitable (not unique) arithmetic subgroup of $E_{7(7)}(\mathbb{Z})$ (see *e.g.* [44, 36, 19, 35], and Refs. therein).

Here we confine ourselves to recall some very basic facts on extremal BHs (for further detail, see *e.g.* [45], and Refs. therein), and then we will mention some recent developments on multi-center solutions.

For simplicity's sake, we consider the particular class of *extremal* BH solutions constituted by static, asymptotically flat, spherically symmetric solitonic objects with dyonic charge vector $\mathcal{Q}$ and scalars ϕ describing trajectories (in the radial evolution parameter r) with[8] *fixed points* determined by the *Attractor Mechanism* [46]:

$$
\begin{cases}
\lim_{r \to r_H^+} \phi(r) = \phi_H(\mathcal{Q}); \\[2ex]
\lim_{r \to r_H^+} \frac{d\phi(r)}{dr} = 0.
\end{cases}
\tag{2.1}
$$

At the horizon, the scalars lose memory of the initial conditions (*i.e.* of the asymptotic values $\phi_\infty \equiv \lim_{r \to \infty} \phi(r)$), and the fixed (attractor) point $\phi_H^a(\mathcal{Q})$ only depends on the BH charges $\mathcal{Q}$. In the supergravity limit, for $\mathcal{N} > 2$ superymmetry, the attractor behavior of such BHs is now completely classified (see *e.g.* [47, 48] for a review and list of Refs.).

The classical BH entropy is given by the Bekenstein-Hawking entropy-area formula [49]

$$
S(\mathcal{Q}) = \frac{A_H(\mathcal{Q})}{4} = \pi V_{BH}(\phi_H(\mathcal{Q}), \mathcal{Q}) = \pi \sqrt{|\mathcal{I}_4(\mathcal{Q})|}.
\tag{2.2}
$$

where V_{BH} is the effective BH potential [50] (see Eq. (2.22) below).

The last step of (2.2) holds[9] for those theories admitting a quartic polynomial invariant $\mathcal{I}_4$ in the (symplectic) representation of the electric-magnetic duality group in which $\mathcal{Q}$ sits. This is the case *at least* for the *"groups of type E_7"* [51], which are the electric-magnetic duality groups of supergravity theories in $d = 4$ with *symmetric* scalar manifolds (see *e.g.* [37] for recent developments, and a list of Refs.). These include all $\mathcal{N} \geqslant 3$ supergravities as well as a broad class of $\mathcal{N} = 2$ theories in which the vector multiplets' scalar manifold is a *special Kähler* symmetric space (see *e.g.* [52, 53, 54, 68], and Refs. therein). In the D-brane picture of type IIA supergravity compactified on Calabi-Yau threefolds CY_3, charges can be denoted by q_0 ($D0$), q_a ($D2$), p^a ($D4$) and p^0 ($D6$), and the quartic invariant polynomial $\mathcal{I}_4$ is given by [55]

$$
\mathcal{I}_4 = -\left(p^0 q_0 + p^a q_a\right)^2 + 4\left(-p^0 \mathcal{I}_3(q) + q_0 \mathcal{I}_3(p) + \frac{\partial \mathcal{I}_3(p)}{\partial p^a} \frac{\partial \mathcal{I}_3(q)}{\partial q_a}\right);
\tag{2.3}
$$

$$
\mathcal{I}_3(p) \equiv \frac{1}{3!} d_{abc} p^a p^b p^c; \quad \mathcal{I}_3(q) \equiv \frac{1}{3!} d^{abc} q_a q_b q_c,
\tag{2.4}
$$

where d_{abc} and d^{abc} are completely symmetric rank-3 invariant tensors of the relevant electric and magnetic charge irreps. of the U-duality group in $d = 5$. A typical (single-center) BPS

---

[8]The subscript "H" denotes the evaluation at the BH event horizon, whose radial coordinate is r_H (see treatment below).

[9]Incidentally, the last step of (2.2) also holds for arbitrary cubic scalar geometry if particular charge configurations are chosen.

configuration is (q_0, p^a), with all charges positive (implying $\mathcal{I}_4 > 0$), while a typical non-BPS configuration is (p^0, q_0) (implying $\mathcal{I}_4 < 0$), see *e.g.* the discussion in [56] (other charge configurations can be chosen as well). In the dressed charge basis, manifestly covariant with respect to the $\mathcal{R}$-symmetry group, the charges arrange into a complex skew-symmetric central charge matrix Z_{AB}. This latter can be skew-diagonalized to the form [57]

$$Z_{AB} = \text{diag}\,(z_1, z_2, z_3, z_4) \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.5}$$

and the quartic invariant can be recast in the following form [58]:

$$\mathcal{I}_4 = \sum_{i=1}^{4} |z_i|^4 - 2 \sum_{i<j=1}^{4} |z_i|^2 |z_j|^2 + 4 \prod_{i=1}^{4} z_i + 4 \prod_{i=1}^{4} \overline{z}_i. \tag{2.6}$$

In such a basis, a typical BPS configuration is the one pertaining to the Reissner-Nördstrom BH, with charges $z_1 = (q + ip)$ and $z_2 = z_3 = z_4 = 0$ (implying $\mathcal{I}_4 = \left(q^2 + p^2\right)^2 > 0$), whereas a typical non-BPS configuration has (at the event horizon) $z_i = \rho e^{i\pi/4}$ $\forall i = 1, ..., 4$ (implying $\mathcal{I}_4 = -16\rho^4 < 0$); see *e.g.* the discussion in [59, 60, 61].

The simplest example of BH metric is the Schwarzschild BH:

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{2.7}$$

where M is the ADM mass [62], and $d\Omega^2 = d\theta^2 + sin^2\theta d\psi^2$. This BH has no *naked singularity*, *i.e.* the singularity at $r = 0$ is *covered* by the event horizon at $r_H = 2M$.

The metric (2.7) can be seen as the neutral $q, p \to 0$ limit of the Reissner-Nordström (RN) BH:

$$ds^2_{RN} = -\left(1 - \frac{2M}{r} + \frac{q^2 + p^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{q^2 + p^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{2.8}$$

Such a metric exhibits two horizons, with radii

$$r_\pm = M \pm \sqrt{M^2 - q^2 - p^2}. \tag{2.9}$$

In the *extremal* case $r_+ = r_-$, and it holds that

$$M^2 = q^2 + p^2, \tag{2.10}$$

thus a unique event horizon exists at $r_H = M$. Notice that for RN BHs the extremality condition coincides with the saturation of the *BPS bound* [63]

$$M^2 \geqslant q^2 + p^2. \tag{2.11}$$

By defining $\rho \equiv r - M = r - r_H$, the extremal RN metric acquires the general static Papapetrou-Majumdar [64] form

$$ds^2_{RN,extr} = -\left(1 + \frac{M}{\rho}\right)^{-2} dt^2 + \left(1 + \frac{M}{\rho}\right)^2 \left(d\rho^2 + \rho^2 d\Omega^2\right) = -e^{2U} dt^2 + e^{-2U} d\vec{x}^2, \tag{2.12}$$

where $U = U(\vec{x})$ is an harmonic function satisfying the $d = 3$ Laplace equation

$$\Delta e^{-U(\vec{x})} = 0. \tag{2.13}$$

In order to determine the near-horizon geometry of an extremal RN BH, let us define a new radial coordinate as $\tau = -\frac{1}{\rho} = \frac{1}{r_H - r}$. Thus, after a further rescaling $\tau \to \frac{\tau}{M^2}$, the near-horizon limit $\rho \to 0^+$ of extremal metric (2.12) reads

$$\lim_{\rho \to 0^+} ds^2_{RN,extr} = \frac{M^2}{\tau^2} \left(-dt^2 + d\tau^2 + \tau^2 d\Omega^2 \right), \tag{2.14}$$

which is nothing but the $AdS_2 \times S^2$ Bertotti-Robinson metric [34], both *flat* and *conformally flat*.

In presence of scalar fields coupled to the BH background, the BPS bound gets modified, and in general extremality does not coincide with the saturation of BPS bound (and thus with supersymmetry preservation) any more. Roughly speaking, the charges $\mathcal{Q}$ gets "dressed" with scalar fields ϕ into the central extension of the local $\mathcal{N}$-extended supersymmetry algebra, which is an antisymmetric complex matrix $Z_{AB}(\phi, \mathcal{Q})$, named *central charge matrix* $(A, B = 1, ..., \mathcal{N})$:

$$\begin{cases} \left\{ \mathbf{Q}_{\alpha A}, \overline{\mathbf{Q}}^B_{\dot{\alpha}} \right\} = \delta^B_A \sigma^\mu_{\alpha\dot{\alpha}} P_\mu; \\[2mm] \left\{ \mathbf{Q}_{\alpha A}, \mathbf{Q}_{\beta B} \right\} = \epsilon_{\alpha\beta} Z_{AB}(\phi, \mathcal{Q}). \end{cases} \tag{2.15}$$

In general

$$Z_{AB}(\phi, \mathcal{Q}) = L^{\mathbb{A}}_{AB}(\phi) \mathcal{Q}_{\mathbb{A}}, \tag{2.16}$$

where $L^{\mathbb{A}}_{AB}(\phi)$ are the scalar field-dependent symplectic sections of the corresponding *(generalized) special geometry* (see *e.g.* [26, 59, 37], and Refs. therein).

In the BH background under consideration, the general *Ansätze* for the vector 2-form field strengths $F^\Lambda_{\mu\nu}$ of the n_V vector fields $(\Lambda = 1, \ldots, n_V)$ and their duals $G_{\Lambda\mu\nu} = \frac{\delta\mathcal{L}}{\delta F^\Lambda_{\mu\nu}}$ are given by [50]

$$F = e^{2U} \mathbb{C}\mathcal{M}(\phi) \mathcal{Q} dt \wedge d\tau + \mathcal{Q} \sin\theta d\theta \wedge d\psi; \tag{2.17}$$

$$F = \begin{pmatrix} F^\Lambda_{\mu\nu} \\ G_{\Lambda\mu\nu} \end{pmatrix} \frac{dx^\mu dx^\nu}{2}, \tag{2.18}$$

and electric and magnetic charges $\mathcal{Q} \equiv \left(p^\Lambda, q_\Lambda \right)^T$ are defined by

$$q_\Lambda \equiv \frac{1}{4\pi} \int_{S^2_\infty} G_\Lambda, \qquad p^\Lambda \equiv \frac{1}{4\pi} \int_{S^2_\infty} F^\Lambda, \tag{2.19}$$

where S^2_∞ is the 2-sphere at infinity. $\mathcal{M}(\phi)$, already discussed in Sec. 1, is a $2n_V \times 2n_V$ real symmetric $Sp(2n_V, \mathbb{R})$ matrix (see Eq. (1.15)) whose explicit form reads [26]

$$\mathcal{M}(\phi) = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}, \tag{2.20}$$

with $I \equiv \text{Im}\mathcal{N}_{\Lambda\Sigma}$ and $R \equiv \text{Re}\mathcal{N}_{\Lambda\Sigma}$, where $\mathcal{N}_{\Lambda\Sigma}$ is the (scalar field dependent) kinetic vector matrix entering the $d = 4$ Lagrangian density

$$\mathcal{L} = -\frac{R}{2} + \frac{1}{2} g_{ij}(\phi) \partial_\mu \phi^i \partial^\mu \phi^j + I_{\Lambda\Sigma} F^\Lambda \wedge^* F^\Sigma + R_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma. \tag{2.21}$$

The black hole effective potential [46] is given by

$$V_{BH}(\phi, \mathcal{Q}) = -\frac{1}{2} \mathcal{Q}^T \mathcal{M}(\phi) \mathcal{Q}, \tag{2.22}$$

9

This is the effective potential which arises upon reducing the general $d \geqslant 4$ Lagrangian on the BH background to the $d = 1$ almost geodesic action describing the radial evolution of the $n_V + 1$ scalar fields $(U(\tau), \phi^i(\tau))$ [65]:

$$S = \int \mathcal{L} d\tau = \int (\dot{U} + g_{ij}\dot{\phi}^i\dot{\phi}^j + e^{2U}V_{BH}(\phi(\tau), p, q)d\tau. \tag{2.23}$$

In order to have the same equations of motion of the original theory, the action must be complemented with the Hamiltonian constraint, which in the extremal case reads [50]

$$\dot{U}^2 + g_{ij}\dot{\phi}^i\dot{\phi}^j - e^{2U}V_{BH}(\phi(\tau), p, q) = 0. \tag{2.24}$$

The black hole effective potential V_{BH} can generally be written in terms of the superpotential $W(\phi)$ as

$$V_{BH} = W^2 + 2g^{ij}\partial_i W \partial_j W. \tag{2.25}$$

This formula can be viewed as a differential equation defining W for a given V_{BH}, and it can lead to multiple choices, one corresponding to BPS solutions, and the others associated to non-BPS ones. W allows to rewrite the ordinary second order supergravity equations of motion

$$\ddot{U} = e^{2U}V_{BH}; \tag{2.26}$$

$$\ddot{\phi}^i = g^{ij}\frac{\partial V_{BH}}{\partial \phi_j}e^{2U}, \tag{2.27}$$

as first order flow equations, defining the radial evolution of the scalar fields ϕ^i and the warp factor U from asymptotic (radial) infinity towards the black hole horizon [66] :

$$\dot{U} = -e^U W, \qquad \dot{\phi}^i = -2e^U g^{ij}\partial_j W. \tag{2.28}$$

At the prize of finding a suitable *"fake"* first order superpotential W, one only has to deal with these first order flow equations even for non-supersymmetric solutions, where one does not have Killing spinor equations [66, 67].

For $\frac{1}{N}$-BPS supersymmetric BHs in $\mathcal{N} \geqslant 2$ supergravity theories (with central charge matrix Z_{AB}), $\mathcal{W}$ is given by the square root[10] $\sqrt{\lambda_h}$ of the largest of the eigenvalues of $Z_{AB}Z^{\dagger BC}$ [66, 67]. Furthermore, $\mathcal{W}$ has a known analytical expression for all $\mathcal{N} \geqslant 2$ charge configurations with $\mathcal{I}_4 > 0$ (for $\mathcal{N} = 2$, this applies to special Kähler geometry based on symmetric spaces, see *e.g.* [68]) [67]. For $\mathcal{I}_4 < 0$, $\mathcal{W}^2$ has an analytical expression for rank-1 and rank-2 cosets [69, 70, 71], while it is known to exist in general as a solution of a sixth order algebraic equation [70, 71, 72].

The Bekenstein-Hawking BH entropy [49] (2.2) can be written in terms of W as follows:

$$S(\mathcal{Q}) = \pi W^2\big|_{\partial W=0}, \tag{2.29}$$

where the critical points of the suitable W reproduce a class of critical points of V itself. It is worth remarking that the value of the superpotential W at radial infinity also encodes other basic properties of the extremal black hole, namely its ADM mass [62], given by ($\phi^i_\infty \equiv \lim_{r\to\infty}\phi^i(r)$)

$$M_{ADM}(\phi_\infty, \mathcal{Q}) = \dot{U}(\tau = 0) = W(\phi_\infty, \mathcal{Q}), \tag{2.30}$$

and the scalar charges

$$\Sigma^i(\phi_\infty, \mathcal{Q}) = 2g^{ij}(\phi_\infty)\frac{\partial W}{\partial \phi^i}(\phi_\infty, \mathcal{Q}). \tag{2.31}$$

---

[10]The subscript "h" stands for "the highest".

Multi-center BHs are a natural extension of single-center BHs, and they play an important role in the dynamics of quantum theories of gravity, such as superstrings and M-theory.

In fact, interesting multi-center solutions have been found for BPS BHs in $d = 4$ theories with $\mathcal{N} = 2$ supersymmetry, in which the *Attractor Mechanism* [46, 50] is generalized by the so-called *split attractor flow* [73]. This name comes from the existence, for 2-center solutions, of a co-dimension one region (named *marginal stability (MS) wall*) in the scalar manifold, where in fact a stable 2-center BH configuration may decay into two single-center constituents, whose scalar flows then separately evolve according to the corresponding attractor dynamics.

The study of these phenomena has recently progressed in many directions. By combining properties of $\mathcal{N} = 2$ supergravity and superstring theory, a number of interesting phenomena, such as split flow tree, entropy enigma, bound state recombination walls, and microstate counting have been investigated (see *e.g.* [74, 75, 76, 77]).

The MS wall is defined by the condition of stability for a marginal decay of a 2-center BH compound solution with charge $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$ into two single-center BHs (respectively with charges $\mathcal{Q}_1$ and $\mathcal{Q}_2$):

$$M\left(\phi_\infty, \mathcal{Q}_1 + \mathcal{Q}_2\right) = M\left(\phi_\infty, \mathcal{Q}_1\right) + M\left(\phi_\infty, \mathcal{Q}_2\right). \tag{2.32}$$

As mentioned, after crossing the MS wall each flow evolves towards its corresponding attractor point, and the classical entropy of each BH constituent follows the Bekenstein-Hawking formula (2.2). It should be noted that the entropy of the original compound (conceived as a *single-center* BH with total charge $\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2$) can be smaller, equal, or larger than the sum of the entropies of its constituents:

$$S\left(\mathcal{Q}_1 + \mathcal{Q}_2\right) \gtreqless S\left(\mathcal{Q}_1\right) + S\left(\mathcal{Q}_2\right). \tag{2.33}$$

For $\mathcal{N} = 2$ BPS compound and constituents in $\mathcal{N} = 2$, $d = 4$ supergravity (in which $Z_{AB} = \epsilon_{AB} Z$), (2.32) can be recast as a condition on the central charge ($Z_i \equiv M\left(\phi_\infty, \mathcal{Q}_i\right)$, $i = 1, 2$, and $Z_{1+2} \equiv Z\left(\phi_\infty, \mathcal{Q}_1 + \mathcal{Q}_2\right) = Z_1 + Z_2$):

$$|Z_1 + Z_2| = |Z_1| + |Z_2|. \tag{2.34}$$

Furthermore, before crossing the MS wall, the relative distance $|\overrightarrow{x_1} - \overrightarrow{x_2}|$ of the two BH constituents with *mutually non-local* charges $\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \neq 0$ is given by [74]

$$|\overrightarrow{x_1} - \overrightarrow{x_2}| = \frac{1}{2} \frac{\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle |Z_1 + Z_2|}{\mathrm{Im}\left(Z_1 \overline{Z_2}\right)}, \tag{2.35}$$

where

$$2\left|\mathrm{Im}\left(Z_1 \overline{Z_2}\right)\right| = \sqrt{4 |Z_1|^2 |Z_2|^2 - \left(|Z_1 + Z_2|^2 - |Z_1|^2 - |Z_2|^2\right)^2}. \tag{2.36}$$

Correspondingly, the 2-center BH has an intrinsic (orbital) angular momentum, given by [74]

$$\overrightarrow{J} = \frac{1}{2} \langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \frac{\overrightarrow{x_1} - \overrightarrow{x_2}}{|\overrightarrow{x_1} - \overrightarrow{x_2}|}. \tag{2.37}$$

Note that when the charge vectors $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are *mutually local* (*i.e.* $\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle = 0$), $|\overrightarrow{x_1} - \overrightarrow{x_2}|$ is not constrained at all, and $J = 0$. Actually, this is always the case for the scalarless case of extremal Reissner-Nördstrom double-center BH solutions in $\mathcal{N} = 2$ *pure* supergravity. Indeed, in this case the central charge simply reads (see also discussion above)

$$Z_{RN}\left(p, q\right) = q + ip, \tag{2.38}$$

and it is immediate to check that the marginal stability condition (2.34) implies $\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle = q_1 p_2 - p_1 q_2 = 0$.

It is here worth observing that $\text{Im}\left(Z_1 \overline{Z_2}\right) = 0$ both describes marginal and *anti-marginal* stability [76]. *Marginal stability* further requires

$$\text{Re}\left(Z_1 \overline{Z_2}\right) > 0 \Leftrightarrow |Z_1 + Z_2|^2 > |Z_1|^2 + |Z_2|^2. \tag{2.39}$$

The other (unphysical) branch, namely

$$\text{Re}\left(Z_1 \overline{Z_2}\right) < 0 \Leftrightarrow |Z_1 + Z_2|^2 < |Z_1|^2 + |Z_2|^2, \tag{2.40}$$

pertains to *anti-marginal stability*, reached for $|Z_1 + Z_2| = ||Z_1| - |Z_2||$.

Eq. (2.35) implies the stability region for the 2-center BH solution to occur for

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \, \text{Im}\left(Z_1 \overline{Z_2}\right) > 0, \tag{2.41}$$

while it is forbidden for $\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \text{Im}\left(Z_1 \overline{Z_2}\right) < 0$. The scalar flow is directed from the stability region towards the instability region, crossing the MS wall at $\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \text{Im}\left(Z_1 \overline{Z_2}\right) = 0$. This implies that the stability region is placed *beyond* the MS wall, and *on the opposite side* of the split attractor flows.

By using the fundamental identities of $\mathcal{N} = 2$ special Kähler geometry in presence of two (mutually non-local) symplectic charge vectors $\mathcal{Q}_1$ and $\mathcal{Q}_2$ (see *e.g.* [73, 78, 59]), one can compute that at BPS attractor points of the centers 1 *or* 2:

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle = -2\text{Im}\left(Z_1 \overline{Z_2}\right) \Rightarrow 2\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \text{Im}\left(Z_1 \overline{Z_2}\right) = -\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle^2 < 0. \tag{2.42}$$

By using (2.35) and (2.42), one obtains $|\vec{x_1} - \vec{x_2}| < 0$: this means that, as expected, the BPS attractor points of the centers 1 *or* 2 do not belong to the stability region of the 2-center BH solution. Furthermore, the result (2.42) also consistently implies:

stability region :

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \, \text{Im}\left(Z_1 \overline{Z_2}\right) = |\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle| \sqrt{4|Z_1|^2 |Z_2|^2 - \left(|Z_1 + Z_2|^2 - |Z_1|^2 - |Z_2|^2\right)^2} > 0; \tag{2.43}$$

instability region :

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \, \text{Im}\left(Z_1 \overline{Z_2}\right) = -|\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle| \sqrt{4|Z_1|^2 |Z_2|^2 - \left(|Z_1 + Z_2|^2 - |Z_1|^2 - |Z_2|^2\right)^2} < 0, \tag{2.44}$$

where a particular case of (2.44), holding at the attractor points, is given by (2.42).

As shown in [77], by exploiting the theory of *matrix norms*, all above results can be extended *at least* to $\mathcal{N} = 2$ non-BPS states with $\mathcal{I}_4 > 0$, as well as to BPS states in $\mathcal{N} > 2$ supergravity.

For two-center BHs, by replacing $|Z|$ with $\sqrt{\lambda_h}$, the generalization of (2.35) *e.g.* to $\mathcal{N} = 8$ maximal supergravity reads

$$|\vec{x_1} - \vec{x_2}| = \frac{|\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle| \sqrt{\lambda_{1+2,h}}}{\sqrt{4\lambda_{1,h}\lambda_{2,h} - (\lambda_{1+2,h} - \lambda_{1,h} - \lambda_{2,h})^2}}, \tag{2.45}$$

where $\lambda_{1+2,h} \equiv \lambda_h\left(\phi_\infty, \mathcal{Q}_1 + \mathcal{Q}_2\right)$ and $\lambda_{i,h} \equiv \lambda_h\left(\phi_\infty, \mathcal{Q}_i\right)$.

Analogously, also result (2.42) can be generalized *e.g.* to suitable states in $\mathcal{N} = 8$ supergravity. Indeed, by exploiting the $\mathcal{N} = 8$ generalized special geometry identities [59] $(\mathbf{Z}_i \equiv Z_{AB}(\phi_\infty, \mathcal{Q}_i))$

$$\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle = -\mathrm{Im}\left(Tr\left(\mathbf{Z}_1 \mathbf{Z}_2^\dagger\right)\right), \qquad (2.46)$$

one can compute that at the $\frac{1}{8}$-BPS attractor points of the centers 1 *or* 2 it holds

$$|\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle| = \sqrt{4\lambda_{h,1}\lambda_{h,2} - (\lambda_{1,h} + \lambda_{2,h} - \lambda_{1+2,h})^2}. \qquad (2.47)$$

Analogously to the $\mathcal{N} = 2$ case treated above, note that $\frac{1}{8}$-BPS attractor points of the centers 1 *or* 2 do not belong to the stability region of the two-center BH solution, but instead they are placed, with respect to the stability region, on the opposite side of the MS wall.

Acknowledgments

References

[1] E. Cremmer and B. Julia, *The SO(8) Supergravity*, Nucl. Phys. **B159**, 141 (1979).

[2] C. M. Hull and N. P. Warner, *The Structure of the Gauged $\mathcal{N} = 8$ Supergravity Theories*, Nucl. Phys. **B253**, 650 (1985). C. M. Hull and N. P. Warner, *The Potentials of the Gauged $\mathcal{N} = 8$ Supergravity Theories*, Nucl. Phys. **B253**, 675 (1985).

[3] C. Hillmann, $E_{7(7)}$ *Invariant Lagrangian of $d = 4$ $\mathcal{N} = 8$ Supergravity*, JHEP **1004**, 010 (2010), arXiv:0911.5225 [hep-th].

[4] G. Bossard, C. Hillmann and H. Nicolai, $E_{7(7)}$ *Symmetry in Perturbatively Quantised $\mathcal{N} = 8$ Supergravity*, JHEP **1012**, 052 (2010), arXiv:1007.5472 [hep-th].

[5] R. Slansky, *Group Theory for Unified Model Building*, Phys. Rep. **79**, 1 (1981).

[6] S. A. Hartnoll, *Lectures on Holographic Methods for Condensed Matter Physics*, Class. Quant. Grav. **26**, 224002 (2009), arXiv:0903.3246 [hep-th].

[7] M. Rangamani, *Gravity & Hydrodynamics : Lectures on Fluid-Gravity Correspondence*, Class. Quant. Grav. **26**, 224003 (2009), arXiv:0905.4352 [hep-th].

[8] B. Biran, F. Englert, B. de Wit and H. Nicolai, *Gauged $\mathcal{N} = 8$ Supergravity and its Breaking from Spontaneous Compactification*, Phys. Lett. **B124**, 45 (1983); erratum ibidem, **B128**, 461 (1983).

[9] M. J. Duff and C. N. Pope, *Kaluza-Klein Supergravity and the Seven Sphere*, Lectures given at September School on Supergravity and Supersymmetry, ICTP Trieste (Italy), 1982.

[10] B. de Wit and H. Nicolai, $\mathcal{N} = 8$ *Supergravity*, Nucl. Phys. **B208**, 323 (1982).

[11] I. Yokota, *Subgroup $SU(8)/\mathbb{Z}_2$ of compact simple Lie group E_7 and non-compact simple Lie group $E_{7(7)}$ of type E_7*, Math. J. Okoyama Univ. **24**, 53 (1982).

[12] P. Aschieri, S. Ferrara and B. Zumino, *Duality Rotations in Nonlinear Electrodynamics and in Extended Supergravity*, Riv. Nuovo Cim. **31**, 625 (2008), arXiv:0807.4039 [hep-th].

[13] M. K. Gaillard and B. Zumino, *Duality Rotations for Interacting Fields*, Nucl. Phys. **B193**, 221 (1981).

[14] P. Di Vecchia, S. Ferrara and L. Girardello, *Anomalies of Hidden Local Chiral Symmetries in Sigma Models and Extended Supergravities*, Phys. Lett. **B151**, 199 (1985).

[15] N. Marcus, *Composite Anomalies in Supergravity*, Phys. Lett. **B157**, 383 (1985).

[16] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, *The Ultraviolet Behavior of $\mathcal{N}= 8$ Supergravity at Four Loops*, Phys. Rev. Lett. **103**, 081301 (2009), arXiv:0905.2326 [hep-th].

[17] R. Kallosh, *On UV Finiteness of the Four Loop $\mathcal{N}= 8$ Supergravity*, JHEP **0909**, 116 (2009), arXiv:0906.3495 [hep-th].

[18] M. Bianchi, S. Ferrara and R. Kallosh, *Perturbative and Non-Perturbative $\mathcal{N}= 8$ Supergravity*, Phys. Lett. **B690**, 328 (2010), arXiv:0910.3674 [hep-th].

[19] M. Bianchi, S. Ferrara and R. Kallosh, *Observations on Arithmetic Invariants and U-Duality Orbits in $\mathcal{N}= 8$ Supergravity*, JHEP **1003**, 081 (2010), ArXiv:0912.0057 [hep-th].

[20] P. Vanhove, *The Critical Ultraviolet Behaviour of $\mathcal{N}= 8$ Supergravity Amplitudes*, arXiv:1004.1392 [hep-th].

[21] L. J. Dixon, *Ultraviolet Behavior of $\mathcal{N}= 8$ Supergravity*, Lectures presented at ISSP 2009, Aug 29 - Sep 7 '09, Erice (Italy), arXiv:1005.2703 [hep-th].

[22] R. Kallosh, *The Ultraviolet Finiteness of $\mathcal{N}= 8$ Supergravity*, JHEP **1012**, 009 (2010), arXiv:1009.1135 [hep-th].

[23] N. Beisert, H. Elvang, D. Z. Freedman, M. Kiermaier, A. Morales and S. Stieberger, *$E_{7(7)}$ Constraints on Counterterms in $\mathcal{N}= 8$ Supergravity*, Phys. Lett. **B694**, 265 (2010), arXiv:1009.1643 [hep-th].

[24] Z. Bern, J. J. Carrasco, L. Dixon, H. Johansson and R. Roiban, *Amplitudes and Ultraviolet Behavior of $\mathcal{N}= 8$ Supergravity*, arXiv:1103.1848 [hep-th].

[25] M. Henneaux and C. Teitelboim, *Dynamics of chiral (selfdual) p-forms*, Phys. Lett. **B206**, 650 (1988).

[26] A. Ceresole, R. D'Auria and S. Ferrara, *The Symplectic Structure of $\mathcal{N}= 2$ Supergravity and its Central Extension*, Nucl. Phys. Proc. Suppl. **46**, 67 (1996), hep-th/9509160.

[27] C. Brunster and M. Henneaux, *The Action of Twisted Self-Duality*, arXiv:1103.3621 [hep-th].

[28] P. S. Howe and U. Lindström, *Higher Order Invariants in Supergravity*, Nucl. Phys. **B181**, 487 (1981).

[29] J. Björnsson and M. B. Green, *5 Loops in 24/5 Dimensions*, JHEP **1008**, 132 (2010), arXiv:1004.2692 [hep-th].

[30] R. Kallosh, $E_{7(7)}$ *Symmetry and Finiteness of* $\mathcal{N}=8$ *Supergravity*, `arXiv:1103.4115` `[hep-th]`.

[31] M. Green, H. Ooguri and J. H. Schwarz, *Nondecoupling of Maximal Supergravity from the Superstring*, Phys. Rev. Lett. **99**, 041601 (2007), `arXiv:0704.0777 [hep-th]`.

[32] N. Arkani-Hamed, F. Cachazo and J. Kaplan, *What is the Simplest Quantum Field Theory?*, JHEP **09**, 016 (2010), `arXiv:0808.1446 [hep-th]`.

[33] T. Banks, *Why I don't Believe* $\mathcal{N}=8$ *SUGRA is Finite*, talk at the Workshop *"Supergravity versus Superstring Theory in the Ultraviolet"*, PennState Univ, PA USA, August 27-30 2009.

[34] B. Bertotti, *Uniform Electromagnetic Field in the Theory of General Relativity*, Phys. Rev. **116**, 1331 (1959). I. Robinson, *A solution of the Maxwell-Einstein equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys. **7**, 351 (1959).

[35] L. Borsten, D. Dahanayake, M. J. Duff, S. Ferrara, A. Marrani and W. Rubens, *Observations on Integral and Continuous* U-*Duality Orbits in* $\mathcal{N}=8$ *Supergravity*, Class. Quant. Grav. **27**, 185003 (2010), `arXiv:1002.4223 [hep-th]`.

[36] L. Borsten, D. Dahanayake, M. J. Duff and W. Rubens, *Black Holes admitting a Freudenthal Dual*, Phys. Rev. **D80**, 026003 (2009), `arXiv:0903.5517 [hep-th]`.

[37] S. Ferrara, A. Marrani and A. Yeranyan, *Freudenthal Duality and Generalized Special Geometry*, `arXiv:1102.4857 [hep-th]`.

[38] M. J. Duff and P. van Nieuwenhuizen, *Quantum Inequivalence of Different Field Representations*, Phys. Lett. **B94**, 179 (1980).

[39] S. M. Christensen, M. J. Duff, G. W. Gibbons and M. Rocek, *Vanishing One-Loop Beta Function in Gauged* $\mathcal{N}>4$ *Supergravity*, Phys. Rev. Lett. **45**, 161 (1980).

[40] M. J. Duff, *Twenty Years of the Weyl Anomaly*, Class. Quant. Grav. **11**, 1387 (1994), `hep-th/9308075`.

[41] L. D. Faddeev and V. N. Popov, *Feynman Diagrams for the Yang-Mills Field*, Phys. Lett. **B25**, 29 (1967).

[42] M. J. Duff and S. Ferrara, *Generalized Mirror Symmetries and Trace Anomalies*, Class. Quant. Grav. **28**, 065005 (2011), `arXiv:1009.4439 [hep-th]`.

[43] T. L. Curtright, *Charge Renormalization and High Spin Fields*, Phys. Lett. **B102**, 17 (1981).

[44] A. Sen, *Arithmetic of* $\mathcal{N}=8$ *Black Holes*, JHEP **1002**, 090 (2010), `arXiv:0908.0039 [hep-th]`.

[45] S. Ferrara, K. Hayakawa and A. Marrani, *Lectures on Attractors and Black Holes*, Fortsch. Phys. **56**, 993 (2008), `arXiv:0805.2498 [hep-th]`.

[46] S. Ferrara, R. Kallosh and A. Strominger, $\mathcal{N}=2$ *extremal black holes*, Phys. Rev. **D52**, 5412 (1995), `hep-th/9508072`. A. Strominger, *Macroscopic entropy of* $\mathcal{N}=2$ *extremal black holes*, Phys. Lett. **B383**, 39 (1996), `hep-th/9602111`. S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, Phys. Rev. **D54**, 1514 (1996), `hep-th/9602136`. S. Ferrara

and R. Kallosh, *Universality of supersymmetric attractors*, Phys. Rev. **D54**, 1525 (1996), `hep-th/9603090`.

[47] L. Andrianopoli, R. D'Auria, S. Ferrara and M. Trigiante, *Extremal Black Holes in Supergravity*, Lect. Notes Phys. **737**, 661 (2007), `hep-th/0611345`.

[48] S. Bellucci, S. Ferrara, R. Kallosh and A. Marrani, *Extremal Black Holes and Flux Vacua Attractors*, Lect. Notes Phys. **755**, 115 (2008), `arXiv:0711.4547 [hep-th]`.

[49] S. W. Hawking: *Gravitational Radiation from Colliding Black Holes*, Phys. Rev. Lett. **26**, 1344 (1971). J. D. Bekenstein: *Black Holes and Entropy*, Phys. Rev. **D7**, 2333 (1973).

[50] S. Ferrara, G. W. Gibbons and R. Kallosh, *Black Holes and Critical Points in Moduli Space*, Nucl. Phys. **B500**, 75 (1997), `hep-th/9702103`.

[51] R. B. Brown, *Groups of Type E_7*, J. Reine Angew. Math. **236**, 79 (1969).

[52] E. Cremmer, J. P. Derendinger, B. de Wit, S. Ferrara, L. Girardello, C. Kounnas and A. Van Proeyen, *Vector Multiplets Coupled to $\mathcal{N}=2$ Supergravity : SuperHiggs Effect, Flat Potentials and Geometric Structure*, Nucl. Phys. **B250**, 385 (1985).

[53] E. Cremmer and A. Van Proeyen, *Classification of Kähler Manifolds in $\mathcal{N}=2$ Vector Multiplet Supergravity Couplings*, Class. Quant. Grav. **2**, 445 (1985).

[54] M. Günaydin, G. Sierra and P. K. Townsend, *Exceptional Supergravity Theories and the Magic Square*, Phys. Lett. **B133**, 72 (1983). M. Günaydin, G. Sierra and P. K. Townsend, *The Geometry of $\mathcal{N}=2$ Maxwell-Einstein Supergravity and Jordan Algebras*, Nucl. Phys. **B242**, 244 (1984).

[55] S. Ferrara and M. Günaydin, *Orbits of Exceptional Groups, Duality and BPS States in String Theory*, Int. J. Mod. Phys. **A13**, 2075 (1998), `hep-th/9708025`.

[56] A. Ceresole, S. Ferrara and A. Marrani, *4d/5d Correspondence for the Black Hole Potential and its Critical Points*, Class. Quant. Grav. **24**, 5651 (2007), `arXiv:0707.0964 [hep-th]`.

[57] L. K. Hua, *On the Theory of Automorphic Functions of a Matrix Variable. I: Geometrical Basis*, Amer. J. Math. **66**, 470 (1944). C. Bloch and A. Messiah, *The Canonical Form of an Antisymmetric Tensor and its Application to the Theory of Superconductivity,* Nucl. Phys. **B39**, 95 (1962). B. Zumino, *Normal Forms of Complex Matrices*, J. Math. Phys. **3**, 1055 (1962).

[58] R. Kallosh and B. Kol, *E_7 Symmetric Area of the Black Hole Horizon*, Phys. Rev. **D53**, 5344 (1996), hep-th/9602014.

[59] S. Ferrara and R. Kallosh, *On $\mathcal{N}=8$ Attractors*, Phys. Rev. **D73**, 125005 (2006), `hep-th/0603247`.

[60] A. Ceresole, S. Ferrara, A. Gnecchi and A. Marrani, *More on $\mathcal{N}=8$ Attractors*, Phys. Rev. **D80**, 045020 (2009), `arXiv:0904.4506 [hep-th]`.

[61] A. Ceresole, S. Ferrara and A. Gnecchi, *5d/4d U-Dualities and $\mathcal{N}=8$ Black Holes*, Phys. Rev. **D80**, 125033 (2009), `arXiv:0908.1069 [hep-th]`.

[62] R. Arnowitt, S. Deser and C. W. Misner: *Canonical Variables for General Relativity*, Phys. Rev. **117**, 1595 (1960).

[63] G.W.Gibbons and C.H.Hull: *A Bogomol'ny bound for general relativity and solitons in $\mathcal{N} = 2$ supergravity*, Phys. Lett. **B109**, 190 (1982).

[64] A. Papapetrou, Proc. R. Irish Acad. **A51**, 191 (1947). S. D. Majumdar, Phys. Rev. **72**, 930 (1947).

[65] P. Breitenlohner, D. Maison and G. W. Gibbons, *Four-Dimensional Black Holes from Kaluza-Klein Theories*, Commun. Math. Phys. **120**, 195 (1988).

[66] A. Ceresole and G. Dall'Agata, *Flow Equations for non-BPS Extremal Black Holes*, JHEP **0703**, 110 (2007), `hep-th/0702088`.

[67] L. Andrianopoli, R. D'Auria, E. Orazi and M. Trigiante, *First Order Description of Black Holes in Moduli Space*, JHEP **0711**, 032 (2007), `arXiv:0706.0712 [hep-th]`.

[68] B. de Wit, F. Vanderseypen and A. Van Proeyen, *Symmetry Structure of Special Geometries*, Nucl. Phys. **B400**, 463 (1993), `hep-th/9210068`.

[69] A. Ceresole, G. Dall'Agata, S. Ferrara and A. Yeranyan, *First Order Flows for $\mathcal{N} = 2$ Extremal Black Holes and Duality Invariants*, Nucl. Phys. **B824**, 239 (2010), `arXiv:0908.1110 [hep-th]`.

[70] G. Bossard, Y. Michel and B. Pioline, *Extremal Black Holes, Nilpotent Orbits and the True Fake Superpotential*, JHEP **1001**, 038 (2010), `arXiv:0908.1742 [hep-th]`.

[71] A. Ceresole, G. Dall'Agata, S. Ferrara and A. Yeranyan, *Universality of the Superpotential for $d = 4$ Extremal Black Holes*, Nucl. Phys. **B832**, 358 (2010), `arXiv:0910.2697 [hep-th]`.

[72] S. Ferrara, A. Marrani and E. Orazi, *Maurer-Cartan Equations and Black Hole Superpotentials in $\mathcal{N} = 8$ Supergravity*, Phys. Rev. **D81**, 085013 (2010), `arXiv:0911.0135 [hep-th]`.

[73] F. Denef, *Supergravity flows and D-brane stability*, JHEP **0008**, 050 (2000), `hep-th/0005049`. F. Denef, B. R. Greene and M. Raugas, *Split attractor flows and the spectrum of BPS D-branes on the quintic*, JHEP **0105**, 012 (2001), `hep-th/0101135`.

[74] B. Bates and F. Denef, *Exact solutions for supersymmetric stationary black hole composites*, `arXiv:hep-th/0304094`.

[75] R. Kallosh, N. Sivanandam, and M. Soroush, *Exact Attractive Non-BPS STU Black Holes*, Phys. Rev. **D74**, 065008 (2006), `hep-th/0606263`. F. Denef and G. W. Moore, *Split States, Entropy Enigmas, Holes and Halos*, `hep-th/0702146`. F. Denef, D. Gaiotto, A. Strominger, D. Van den Bleeken and X. Yin, *Black Hole Deconstruction*, `hep-th/0703252`. F. Denef and G. W. Moore, *How many black holes fit on the head of a pin?*, Gen. Rel. Grav. **39**, 1539 (2007), `arXiv:0705.2564 [hep-th]`. D. Gaiotto, W. W. Li and M. Padi, *Non-Supersymmetric Attractor Flow in Symmetric Spaces*, JHEP **0712**, 093 (2007), `arXiv:0710.1638 [hep-th]`. M. C. N. Cheng and E. P. Verlinde, *Wall Crossing, Discrete Attractor Flow, and Borcherds Algebra*, SIGMA **4**, 068 (2008), `arXiv:0806.2337 [hep-th]`. E. G. Gimon, F. Larsen and J. Simon, *Constituent Model of Extremal non-BPS Black Holes*, JHEP **0907**, 052 (2009), `arXiv:0903.0719 [hep-th]`. A. Castro and J. Simon, *Deconstructing the D0-D6 system*, JHEP **0905**, 078 (2009), `arXiv:0903.5523 [hep-th]`. J. R. David, *On walls of marginal stability in $\mathcal{N} = 2$ string theories*, JHEP **0908**, 054 (2009), `arXiv:0905.4115 [hep-th]`. J. Manschot, *Stability and Duality in $\mathcal{N} = 2$ Supergravity*, Commun. Math. Phys. **299**, 651 (2010), `arXiv:0906.1767 [hep-th]`. P. Galli

and J. Perz, *Non-supersymmetric extremal multicenter black holes with superpotentials*, JHEP **1002**, 102 (2010), `arXiv:0909.5185 [hep-th]`. A. Sen, *Walls of Marginal Stability and Dyon Spectrum in $\mathcal{N}=4$ Supersymmetric String Theories*, JHEP **0705**, 039 (2007), `hep-th/0702141`. A. Sen, *Two Centered Black Holes and $\mathcal{N}=4$ Dyon Spectrum*, JHEP **0709**, 045 (2007), `arXiv:0705.3874 [hep-th]`. A. Sen, *$\mathcal{N}=8$ Dyon Partition Function and Walls of Marginal Stability*, JHEP **0807**, 118 (2008), `arXiv:0803.1014 [hep-th]`. A. Sen, *Wall Crossing Formula for $\mathcal{N}=4$ Dyons: A Macroscopic Derivation*, JHEP **0807**, 078 (2008), `arXiv:0803.3857`. S. Ferrara, A. Marrani and E. Orazi, *Split attractor Flow in $\mathcal{N}=2$ Minimally Coupled Supergravity*, Nucl. Phys. **B846**, 512 (2011), `arXiv: 1010.2280 [hep-th]`. J. Manschot, B. Pioline and A. Sen, *Wall Crossing from Boltzmann Black Holes*, `arXiv:1011.1258 [hep-th]`. S. Ferrara, A. Marrani, E. Orazi, R. Stora and A. Yeranyan, *Two-Center Black Holes Duality-Invariants for stu Model and its lower-rank Descendants*, `arXiv: 1011.5864 [hep-th]`. L. Andrianopoli, R. D'Auria, S. Ferrara, A. Marrani and M. Trigiante, *Two-Centered Magical Charge Orbits*, `arXiv.1101.3496 [hep-th]`.

[76] E. Andriyash, F. Denef, D. L. Jafferis and G. W. Moore, *Wall-crossing from supersymmetric galaxies*, `arXiv:1008.0030 [hep-th]`. E. Andriyash, F. Denef, D. L. Jafferis and G. W. Moore, *Bound state transformation walls*, `arXiv:1008.3555 [hep-th]`.

[77] S. Ferrara and A. Marrani, *Matrix Norms, BPS Bounds and Marginal Stability in $\mathcal{N}=8$ Supergravity*, JHEP **1012**, 038 (2010), `arXiv:1009.3251 [hep-th]`.

[78] S. Bellucci, S. Ferrara and A. Marrani, *On some properties of the attractor equations*, Phys. Lett. **B635**, 172 (2006), `hep-th/0602161`.